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Contributions to the Development of Mathematical Thinking through Geometric Modeling and Problem Solving

Contribuciones al desarrollo del pensamiento matemático mediante el modelado geométrico y la resolución de problemas.

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Contributions to the Development of Mathematical Thinking through Geometric Modeling and Problem Solving

Abstract

The development of mathematical thinking plays a crucial role in problem-solving and significantly contributes to individual and social progress. This article presents the results of a research project that aimed to contribute to the development of mathematical thinking through geometric modeling and problem-solving in eighth-grade students at a public school in the department of Cundinamarca. Analysis of the PISA and SABER test results revealed the low performance of Colombian students in problem-solving and modeling. Furthermore, the results obtained from the implementation of two exploratory activities demonstrated limited skills in these two processes. Within this framework, eight didactic activities with advanced mathematical content, adapted to the students' ages and educational level, were designed and implemented. These activities integrated two fundamental processes: geometric modeling and problem-solving, as well as two processes intrinsic to these fundamental processes: mathematical visualization and Sense Making . A qualitative approach was used with a participatory action research design, employing eight rubrics, video and audio recordings, participant observation, and a final survey. The results demonstrate that students made significant progress in developing problem-solving skills in simulated and real-world contexts, supported by geometric modeling and mathematical visualization, where they constructed robust mathematical concepts, giving them mathematical meaning and significance. It is concluded that the organization and planning of practical mathematics instruction based on the four processes significantly contributed to the robust development of students' mathematical thinking.

Keywords: Secondary Education, Geometric Modeling, Problem Solving, Sense Making , Activity System, Mathematical Visualization

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Palabras clave: Educación secundaria, modelado geométrico, resolución de problemas, comprensión, sistema de actividades, visualización matemática.

Resumen

El desarrollo del pensamiento matemático desempeña un papel crucial en la resolución de problemas y contribuye significativamente al progreso individual y social. Este artículo presenta los resultados de un proyecto de investigación que buscó contribuir al desarrollo del pensamiento matemático a través del modelado geométrico y la resolución de problemas en estudiantes de octavo grado de una escuela pública del departamento de Cundinamarca. El análisis de los resultados de las pruebas PISA y SABER reveló el bajo desempeño de los estudiantes colombianos en resolución de problemas y modelado. Además, los resultados obtenidos de la implementación de dos actividades exploratorias demostraron habilidades limitadas en estos dos procesos. Dentro de este marco, se diseñaron e implementaron ocho actividades didácticas con contenido matemático avanzado, adaptadas a la edad y el nivel educativo de los estudiantes. Estas actividades integraron dos procesos fundamentales: el modelado geométrico y la resolución de problemas, así como dos procesos intrínsecos a estos: la visualización matemática y la construcción de significado. Se utilizó un enfoque cualitativo con un diseño de investigación-acción participativa, empleando ocho rúbricas, grabaciones de video y audio, observación participante y una encuesta final. Los resultados demuestran que los estudiantes progresaron significativamente en el desarrollo de habilidades para la resolución de problemas en contextos simulados y reales, con el apoyo de modelos geométricos y visualización matemática. De esta manera, construyeron conceptos matemáticos sólidos, dotándolos de significado y relevancia. Se concluye que la organización y planificación de la instrucción matemática práctica, basada en los cuatro procesos, contribuyó significativamente al desarrollo integral del pensamiento matemático de los estudiantes.

Introduction

The development of mathematical thinking through a robust teaching and learning process allows for significant cognitive advancements in middle school students [1]. This process fosters the acquisition of fundamental skills and abilities for studying, understanding, and using mathematical concepts, and for establishing connections and relationships between them. It also contributes to the construction of arguments to validate and represent objects abstractly, enhances the communication of mathematical ideas, and stimulates creativity [2]. Furthermore, strong mathematical thinking significantly impacts daily decision-making, academic and professional success, improves our understanding of the world around us, and prepares us for complex mathematical challenges [3].

Reports from the Programme for International Student Assessment (PISA) [4], which involved 15-year-old students from various educational institutions in Colombia, show that the country's mathematics scores are below the average of participating countries belonging to the Organisation for Economic Co-operation and Development (OECD). This analysis reveals that only a minimal percentage of students can model situations and develop strategies to solve mathematical problems. Similar results are shown in the standardized external assessments administered to all primary and secondary schools in the country (SABER). This situation is due, among other things, to students' limited ability to use modeling during the problem-solving process [5], [6] and their limited mastery of prior knowledge.

The literature review revealed some difficulties students face in the problem-solving process. These include the recognition of variables and the relationships between them [7], limited strategies for analyzing, justifying, understanding, and contextualizing a problem [8], and reduced abilities to explain the relationships between real-world objects and mathematics in order to explore a social problem and address it mathematically [5] and [9]. Therefore, to contribute to improving student performance in the country's educational institutions, a significant change in the teaching and learning process of mathematics is needed, positively impacting the development of students' mathematical thinking. To achieve optimal learning and improved skills, students must demonstrate mastery in problem-solving, supported by the process of geometric modeling through "a methodology that integrates both processes in all mathematical activity" [10]. This integration allows for the timely development of mathematical knowledge and fosters an interest in learning.

Therefore, this research aims to contribute to the development of mathematical thinking in eighth-grade students at a public school in the municipality of Supatá, Cundinamarca, through the robust construction of mathematical concepts. To this end, a system of didactic activities is implemented that includes advanced mathematical content adapted to their grade level. Furthermore, two fundamental processes are integrated into the classroom: geometric modeling and problem-solving in authentic, real-world contexts relevant to the students, along with two complementary processes: mathematical visualization and

Sense Making .

This article presents the interventions and results obtained from the process with 30 students. Before the practical intervention, two exploratory activities were implemented to characterize the students' cognitive performance. Using the collected data, eight didactic activities were designed, encompassing four phases in their implementation: motivation, exploration, structuring, and transfer. A rubric was also developed for each activity to assess performance, and these rubrics were validated by a group of experts. The development of both the didactic activities and the rubrics drew upon the contributions of four theoretical frameworks.

The first point of reference is geometric modeling. According to the research findings of [11], [1], and [12], modeling is the means to connect mathematics and reality, fostering in students an interest in mathematical topics within a specific context. Thus, modeling can be seen as a mediating factor in the teaching and learning of mathematics in schools, due to its connections with reality. According to [11], it allows the use of mathematical language to quantify real-world phenomena and analyze behaviors.

Recognizing that geometric modeling is inherent to mathematical modeling, some definitions presented by authors who have linked geometric modeling to the educational practice of mathematics are recorded. Thus, [13], [14], [15], [16], and [17] consider it to be a process where knowledge of geometry is used to represent objects of reality, where they conceive of "representation" as a fundamental axis of modeling and emphasize its importance for problem-solving.

Therefore, the importance of geometric modeling in schools lies in the fact that it can also be used as a teaching strategy, because it allows for the creation or use of models that facilitate problem-solving and problem-solving in a real or simulated context, giving meaning to the usefulness of mathematics in an enriched learning environment. In this study, geometric modeling is established as an “intentional process of representation, where knowledge and resources are used to carry out the analysis, formulation, and development of a model, which provides elements for making inferences that favor the solution of a problem in a geometric context” [18].

In this regard, [19] recognize that geometric modeling optimizes visual and logical thinking, as well as heuristic skills, which contribute to strengthening students' mathematical thinking. According to [20], mathematical thinking is strengthened through geometric modeling when addressing challenging problems that move from real-world situations to mathematical ones. Furthermore, constructing models and solving problems strengthens problem-solving skills, which benefit abstract reasoning, allowing students to connect informal and formal mathematical knowledge for improved learning and greater progress in their performance levels [21].

The second reference point is problem-solving, which strengthens the teaching and learning process of mathematics in the classroom. For their part, [22] state that problems have occupied a fundamental place in the mathematics curriculum, although the terms "problem" and "problem-solving" have been understood in various ways. Among these meanings, the authors point out the following: "Solving problems as a context, as a skill, and for doing mathematics."

Based on the definitions of challenging problems addressed by [23], [24], [25], [26], [27], and [28], challenging problems are those that: require the integration of coherent concepts and whose solution compels students to construct conceptual networks or maps that promote learning. According to [29], solving challenging problems ensures that students think and reason productively. Therefore, mathematical thinking is expanded and enhanced in this process when geometric modeling is involved. Likewise, according to [2], both geometric modeling and problem-solving allow students to explore their creativity and strengthen problem-solving strategies, because they reinforce skills in concept building to achieve robust mathematical thinking.

The third theoretical framework is mathematical visualization, conceived as a skill that presents visual information in diverse ways, making use of mathematical concepts, language, and gestures. According to [30], [31], and [32], visualization should be linked to all modes of mathematical thinking and forms of representation to reproduce ideas symbolically, numerically, and graphically. Therefore, visualization implies an understanding that stems from intuition through images formed by the mind. Thus, visualization puts cognitive structures into practice for problem-solving because it is a set of mental processes that occur in mathematical activity, allowing for three fundamental actions: representing, transforming, and communicating [18].

The fourth reference point is Sense Making . According to [33], Sense Making , as the creation of meaning, involves the construction of plausible understandings and meanings. It is a very useful activity that requires navigating between heuristics and generalization. In this way, creating meaning can be seen as a process in which a person gives meaning to experience and is able to acquire disciplinary knowledge. For [34], Sense Making in education allows students to frame their activity as the path to constructing new knowledge. Furthermore, it allows them to discover new learning opportunities using their own ideas, intuitions, and experiences. To this extent, mathematical meaning is constructed from addressing problems in diverse contexts as a source of meaning because the student connects elements of their environment and reality.

Materials and methods

The research was conducted using a qualitative approach with a participatory action research design, focusing on the subjects in a comprehensive and holistic way. This involved an inductive inquiry process, interaction with the participants, and analysis of the descriptive data collected as events unfolded. Furthermore, the aim was to construct reality as it was observed, objectively throughout the process, without altering or imposing anything. This allowed us to appropriate an epistemic reality in relation to the participating individuals [34], with the objective of changing reality by offering an opportunity to improve and strengthen the construction of mathematical knowledge in the participating students.

This study was conducted with 30 eighth-grade secondary school students from the Nuestra Señora de la Salud Educational Institution (Supatá, Colombia). This unit of analysis was intentionally selected based on specific group characteristics, the researchers' willingness to engage them, and the students were organized into 10 equal groups, named G1, G2, and so on, up to G10.

Eight didactic activities were implemented, each comprising four phases: exploration, during which mental visualization exercises of geometric objects were developed; motivation, during which simple experiments were conducted using structured and unstructured concrete materials; structuring, where challenging problems were proposed based on simulated contexts; and transfer, where challenging problems were presented in authentic, real-world contexts (places familiar to the students). The problem-solving process was studied based on the three phases proposed by [36]: approach, attack, and review. The modeling process followed the simplified cycle proposed by [37] (simplify, mathematize, interpret, and validate).

Eight rubrics were used to objectively and critically evaluate the learning and skills developed by students during the process. A survey with closed and open-ended questions was administered to identify students' perceptions of the classroom activities. Additionally, video recordings were analyzed and participant observation was conducted to identify arguments related to the achievements and difficulties in the learning process.

Results and Discussion

The analysis of the results demonstrates effective mastery of the mathematical content covered. Students utilize their prior knowledge, not only constructing new understanding but also strengthening areas they had not yet fully mastered, which aligns with the findings of [21]. Although the topics covered are not part of the curriculum and are at an advanced level, students demonstrate a solid grasp of them and are motivated to explore and consolidate new learning (Table I). The planning and execution of activities in the four stages (motivation, exploration, structuring, and transfer) show remarkable cognitive progress on the part of the students.

Table I Learning achievements attained by students in each didactic activity.

Activity No.	Aim	Learning achievements
1 **Hamiltonian Way**	To build the concept of the Hamiltonian path through an authentic real-world context.	L1. They used the concept of path to find routes and patterns in situations involving connections between points or places. L2. They identified the characteristics of a Hamiltonian path L3. They recognized Hamiltonian paths in a real context.

2 **Hamiltonian Cycle**	To build the concept of the Hamiltonian cycle through an authentic real-world context.	L1. They understood how a route can begin and end at the same vertex, forming a closed cycle. L2. They established differences between a path and a Hamiltonian cycle based on visual representations.
3 **Triangulation of polygons**	Finding the area of two-dimensional figures through decomposition into triangles by a maximum set of diagonals.	L1. They analyzed how triangles can form a polygon with a larger area. L2. They broke down figures into smaller parts and reconstructed the original figure by adding up the areas of the triangles.
4 **Chvatal's Theorem**	Use triangulation of the area of a two-dimensional figure and coloring of its vertices to represent statements of Chvatal's art gallery theorem.	L1: They decomposed the area of a polygon into areas of triangles. L2: They determined the chromatic number where they explore structural properties of graphs L3: They recognized that a polygon with n vertices can be monitored using a maximum of [n/3] cameras.
5 **Voronoi diagrams**	Identify the concepts of point, plane, half-plane, segment, perpendicular bisector, and circumcenter in the construction of a Voronoi diagram .	L1. They represented the division of a space into regions based on proximity to a set of specific points. Voronoi regions as a result of dividing space based on distance to points.
6 **Viviani's Theorem (1)**	Recognize the characteristics from the possible locations of a point in an equilateral triangular region for the visualization of statements made in Viviani's theorem.	L1. They identified that the sum of the distances from a point to each of the sides is equal to the height of the triangle. L2. They recognized that regardless of the position of the point within the equilateral triangle, the sum of the distances to the sides will always be equal to the height of the triangle.
7 **Viviani's Theorem (2)**	Recognize the given characteristics based on the possible locations of a point in a quadrangular region.	L1. They recognized that regardless of the position of the point within a square, the sum of the distances to the sides will always be equal to the semiperimeter. L2. They applied mathematical concepts to solve real-world and simulated situations.
8 **Carpet Theorem**	Finding the congruence of two-dimensional figures through area decomposition to visualize statements made in the carpet theorem.	L1. They identified how two carpets overlap and how the floor is divided into distinct regions. L2. They understood the concept of surface and how a surface can be divided into different parts using geometric objects. L3. They calculated areas of overlapping geometric figures.

Regarding visualization, the groups are able to quickly and clearly grasp the ideas behind the proposed problems and develop skills to carry out various forms of representation, as [29] suggests. Likewise, they can verify whether their solutions or those of their peers make sense and how to communicate them effectively, as a key part of using metacognitive strategies that students rely on to solve mathematical problems, as [3] suggests. This is because potential learning arises from interaction and socialization with peers.

Regarding problem-solving in the "approach" phase, students identify strategies for solving problems, making use of the experiential aspect that occurs in the initial moments of the activity. In the "attack" phase, they rely on geometric rules, construct models to arrive at the solution, and in the "revision" phase, they recognize their successes and discover the various ways to solve them. In this sense, students potentially acquire problem-solving skills as proposed by [35], since these skills are fostered as strategies are consolidated in each phase, allowing mathematical problems to become more understandable and easier to solve.

Focusing on the problem-solving phases, 72% of the groups achieved a high performance level (HD) in the approach phase and 28% in the medium performance (MP). In the attack phase, 73% achieved HD and 27% MP, and in the review phase, 81% achieved HD and 19% MP. Significant progress was observed in each phase. The most challenging phase for the groups was the "approach" phase, due to their limited skills in understanding a problem and identifying variables. This improved with classroom work, as shown in Figure 1.

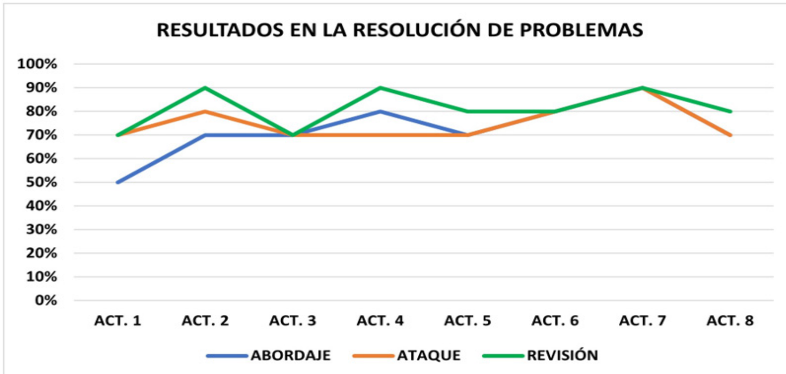

Figure 1 Group performance in the Problem Solving process.

During the development of the activities, problem-solving is a dynamic process that stimulates students' creativity, demonstrating the assumptions of [2], where they apply acquired knowledge to give meaning to reality. In this process, students' ability to analyze, understand, reason, and generalize from diverse contexts is evident, drawing on the principles outlined in [29]. Therefore, solving challenging mathematical problems integrates mathematical modeling, with both processes developing inherently within the student. Thus, problem-solving involves situations in real or simulated contexts that allow for determining the reliability of the model and the dependability of the answers obtained. Regarding geometric modeling, in the "simplify" phase, the groups understand the problems through heuristic questioning and identify key aspects. In the "mathematize" phase, they create various models and use mathematical algorithms to approach the solution. In the "interpret" phase, based on the results obtained by each group and through sharing with their peers, they consolidate the concepts planned for each activity. In this sense, the groups relate these interpretations to the real-world context of the problem, and in the "validate" phase, the students review their answers, identifying whether they are correct and whether they provide a solution to the problem. They also clearly communicate their results and how they arrived at them to their classmates. In this way, the students successfully complete the four phases of modeling, achieving considerable problem-solving skills, as discerned in [37].

In the phases of geometric modeling, the evolution is relevant as implementation progresses. In the simplification phase, 70% of the groups achieved competency development at a DA level and 30% at a DM level. In the mathematization phase, 73% reached a DA level and 27% a DM level. In the interpretation phase, 75% achieved a DA level and 25% a DM level. In the validation phase, 81% were at a DM level and 19% at a DA level. Consequently, the simplification and mathematization phases were the most demanding for the groups, as shown in Figure 2 .

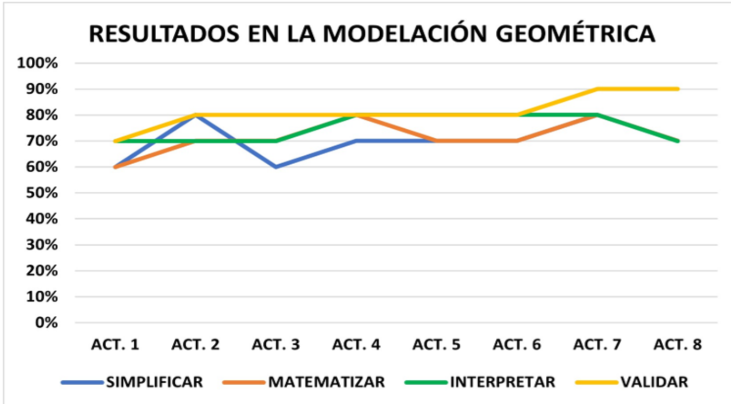

Figure 2 Student performance in the Geometric Modeling process

Geometric modeling contributes substantially to the understanding of mathematics, giving meaning to simplified reality, which students can then transform into mathematical formality, as [19] acknowledges. The geometric models developed establish close links with problem-solving, meaning, and mathematical thinking. Therefore, problem-solving promotes learning and continuous progress through the construction of models for analysis and obtaining a solution, which fosters experience and allows for their application to similar situations.

Consequently, geometric modeling and problem-solving offer students the ability to recognize the importance and benefits of mathematics, improving motivation and understanding, as demonstrated by [2]. Furthermore, it allows them to select and appropriately use mathematical tools in both real and simulated environments. It also facilitates the connection between theory and practice, fostering meaningful learning that contributes to a solid understanding of mathematical concepts.

In Sense Making , the groups develop a meaningful understanding of geometric concepts, recognizing the claims of [34]. They also approach generalizing rules from situations to strengthen their mathematical sense, in accordance with [3], and develop skills in solving challenging problems in context, fostering the ability to visualize and conceptualize geometric relationships.

Regarding the overall results of the eight activities, in each of the five processes, students achieved the proposed learning outcomes and reached the medium and high levels. This demonstrates that the processes are strengthened as classroom interventions continue. Therefore, the average results show that 62% of groups are at the DA level and 38% at the DM level, as shown in Figure 3.

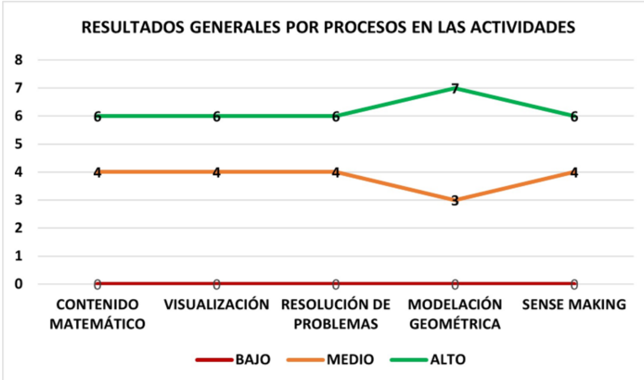

Figure 3 Overall student results in the 8 activities

The survey conducted with students after the classroom intervention reveals that 70% of students recognize a high degree of motivation for constructing their own learning, 80% believe the activities motivated them to work independently, 90% state that the problems addressed in class are challenging and allow them to construct new mathematical concepts, and 70% identify improvements in their skills and abilities in problem-solving, geometric modeling, visualization, and Sense Making . To this extent, there is a good consensus among the working groups and positive validation of the teaching activities implemented.

Regarding the open-ended question about the aspects they found interesting in the activities, the groups offered diverse perspectives. Among these, a student from group 3 (G3) stated that "The problems were interesting. Although some were difficult, we were able to solve them." Meanwhile, G7 affirmed that "It was interesting to use the computer and software to solve the problems," among other comments that demonstrate the positive impact the development of the didactic activities had on student learning and the motivation generated during the intervention.

Conclusions

The proposed set of activities enhanced the learning process, facilitating the development of cognitive skills, abilities, and the comprehensive construction of meaning in concepts. This enabled the student to successfully integrate technology, effective communication, creativity, and teamwork into their learning.

Students acquired problem-solving skills, learning to implement strategies to identify relevant information in each problem. They were also able to break down problems into simpler components, applying mathematical concepts to real or simulated contexts, and improved their communication skills by clearly expressing mathematical ideas.

During the geometric modeling phase, students tackled mathematical problems by applying geometric principles. They planned and built valid models that enabled them to reflect on real-world situations, simplifying complexities into more accessible geometric models in pursuit of abstraction and generalization. Exchanging ideas with their peers enriched their understanding, and they showed progressive improvement as they advanced in implementing the system of activities.

The use of mathematical visualization proved beneficial to the learning process in implementing the system of activities. Progress was also made in strengthening skills to identify relationships between geometric objects and figures in space. Through mental visualization activities during the motivation stage, students were able to imagine and manipulate objects and geometric figures.

In creating mathematical sense and meaning, students made remarkable progress by establishing connections between mathematical concepts and their application in diverse contexts. They also improved their problem-solving skills and their ability to apply modeling in real-world or simulated situations, expressing their ideas in mathematical terms. Furthermore, they were able to communicate their ideas effectively and were aware of their progress in the learning process.

The development of activities at each stage (motivation, exploration, structuring, and transfer) is considered a significant contribution to learning. Therefore, the organization and planning of practical mathematics instruction significantly contributed to the robust development of students' mathematical thinking.

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